

## Precise Calculation of the Dynamical Exponent of Two-Dimensional Percolation

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We report numerical data obtained on the special-purpose computer PERCOLA for the exponent  $\tilde{\nu}$  of the electrical conductivity of 2D percolation. The extrapolation yields  $\tilde{\nu} = 0.9745 \pm 0.0015$  and a correction to the scaling exponent  $\omega = 1.2 \pm 0.2$ .

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**KEY WORDS:** Dynamical exponent; percolation; electrical conductivity; strip method; special-purpose computer.

Transport phenomena on fractals, especially on the percolation cluster at the threshold, have recently been studied with much detail (see ref. 1 for review). It has become clear that phenomena such as anomalous diffusion, electrical conductivity, or the phonon spectrum are all dominated by one dynamical exponent. In the case of electrical conductivity this exponent  $\tilde{\nu}$  describes how the resistivity  $R$  diverges with the linear size  $n$  of the system,  $R \sim n^{\tilde{\nu}}$ . In two dimensions this exponent happens to agree, because of duality, with the exponent  $\tilde{\nu}$  that determines how  $R$  vanishes with  $n$  if the cluster is superconducting.

The numerical value of this dynamical exponent has been the subject of substantial controversy. Based on various arguments,  $\tilde{\nu} = 91/96 = 0.948^{(2)}$  and  $\tilde{\nu} = 1^{(3)}$  have been proposed for two-dimensional percolation. Subsequent numerical work using different methods has supported values of  $\tilde{\nu}$  between these two conjectures<sup>(4,5)</sup> and excluding them both from their error

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bars. In the meantime, however, new arguments favoring the above conjectures were put forward<sup>(6)</sup> and doubts about the size of the numerical error bars have been occasionally raised. In order to shed light upon this issue, we performed another numerical determination in 2D of  $\tilde{s} = \tilde{t}$  applying the method of ref. 4, but investing an at least 500 times larger effort in computation, using the special-purpose computer PERCOLA.<sup>(7,8)</sup>

The method<sup>(4,9)</sup> consists in calculating exactly the electrical resistance of a strip of width  $n$  and length  $L$  where a fraction  $p$  of bonds is superconducting ( $\rho = 0$ ) and the complementary fraction  $1 - p$  is normal conducting ( $\rho = 1$ ). For bond percolation this fraction  $p$  is just the bond percolation threshold  $p_c$  and for site percolation it is given by the probability that two neighboring sites are both occupied, each individual site being occupied with the critical probability of site percolation. In the transverse direction we impose periodic boundary conditions. In the longitudinal direction the  $n$  currents  $I_i$  and potential drops  $V_i$  that are measured at each of the  $n$  left endpoints are related through

$$V_i = \sum_{j=1}^n R_{ij} I_j \quad (1)$$

while all the right endpoints are grounded. The resistance matrix  $\{R_{ij}\}$  is calculated iteratively by adding bond after bond. It is updated according to

$$R'_{ij} = R_{ij} + \rho \delta_{ix} \delta_{xj} \quad (2)$$

for a longitudinal bond at line  $\alpha$  and according to

$$R'_{ij} = R_{ij} - \frac{(R_{ix} - R_{i\beta})(R_{\alpha j} - R_{\beta j})}{\rho + R_{\alpha\alpha} + R_{\beta\beta} - R_{\alpha\beta} - R_{\beta\alpha}} \quad (3)$$

provided  $R_{\alpha\alpha} + R_{\beta\beta} - R_{\alpha\beta} - R_{\beta\alpha}$  is nonzero, otherwise  $R'_{ij} = R_{ij}$ , for a transverse bond between lines  $\alpha$  and  $\beta$ . After many iterations the resistance per length  $R_{11}(n/L)$  converges toward its limit value  $R_n$  for an infinite strip as  $L^{-1/2}$ . For large  $n$ ,  $R_n$  should vanish like  $n^{-3}$  and in two dimensions  $\tilde{t} = \tilde{s}$  because of duality.<sup>(10)</sup>

The calculation of  $R_{ij}$  via (2) and (3) was performed on the special-purpose computer PERCOLA,<sup>(7)</sup> a 64-bit floating-point processor with an architecture optimized for calculations of the type given by (3). Random numbers  $X_k$  generated through the lagged-Fibonacci sequence<sup>(11)</sup>

$$X_k = (X_{k-r} + X_{k-s}) \bmod 2^{32} \quad (4)$$

are calculated in parallel. PERCOLA is microprogrammable and turns at 25 Mflops. For our type of calculation it is about 10% faster than the

vectorized program run on one processor of the Cray X-MP. Memory allows the calculation of strips of width  $n \leq 255$  in two dimensions. Technical details about the machine are given elsewhere.<sup>(8)</sup>

All our calculations were done on the square lattice. We considered bond percolation at  $p_c = 0.5$  and site percolation at  $p_c = 0.592745$  (2).<sup>(12)</sup> Our results are presented in Table I. The lengths  $L$  of the strips are at least 100 times longer and the statistical error bars  $\Delta R_n$ , at least ten times smaller than in ref. 4. One hundred days of CPU time is needed to obtain these data, 50 days alone for the two values for  $n = 40$ , since the time required grows with  $Ln^3$ .

While generating the above data we encountered an unexpected problem that turned out to be due to the random numbers. Using the generator  $(r, s) = (17, 5)$  of (4), which had been implemented also on another special-purpose computer,<sup>(13)</sup> we observed for bond percolation at  $n = 9$  a spurious deviation of the data from the expected monotonic behavior by about ten times the size of the error bars (independent of the seed). We also encountered weaker, significant deviations at some other sizes (e.g.,  $n = 5$ ) and also for the generators  $(r, s) = (31, 13)$  at  $n = 10$  and

**Table I. Resistance  $R_n$  per Length for Bond and Site Percolations, Its Statistical Mean Square Deviation  $\Delta R_n$ , and the Length  $L$  of the Strip As a Function of the Strip Width  $n$**

$n$	Bond			Site		
	$R_n$	$\Delta R_n \times 10^{-6}$	$L$	$R_n$	$\Delta R_n \times 10^{-6}$	$L$
2	0.274836	41	1.40E + 09	0.455632	27.6	1.00E + 09
3	0.176294	35	9.92E + 08	0.328978	17.8	1.00E + 09
4	0.128238	50	1.02E + 09	0.255901	20.3	1.00E + 09
5	0.101079	23	1.02E + 09	0.208946	15.8	1.00E + 09
6	0.083692	23	9.60E + 08	0.176553	19.5	1.00E + 09
7	0.071592	27	1.02E + 09	0.152897	19.8	1.00E + 09
8	0.062579	18	1.02E + 09	0.134827	26.8	1.00E + 09
9	0.055687	21	9.60E + 08	0.120639	14.9	1.00E + 09
10	0.050185	16	1.02E + 09	0.108915	23.9	1.00E + 09
11	0.045672	19	1.02E + 09	0.099313	17.8	1.00E + 09
12	0.041936	22	1.02E + 09	—	—	—
15	0.033609	13	1.02E + 09	0.073930	12.1	1.00E + 09
17	0.029740	16	1.02E + 09	—	—	—
20	0.025237	40	7.68E + 08	0.056381	11.6	1.00E + 09
21	0.024153	10	9.28E + 08	—	—	—
25	0.020355	69	2.60E + 07	—	—	—
40	0.012845	5	6.00E + 08	0.028737	12.7	5.10E + 08

$(r, s) = (55, 24)$  at  $n = 20$ . Although we do not understand the origin of these flaws, they are apparently inherent to the family of generators of (4). These problems could only be observed because of the long sequences (up to  $10^{11}$ ) that we generated; still, the periods of the three generators are respectively larger than  $10^{14}$ ,  $10^{18}$ , and  $10^{25}$ . We note that in other cases where sequences of this length have been used, similar problems were observed.<sup>(14)</sup> Because of this problem we were forced to repeat our calculation for a large part of the values of Table I with up to three different generators in order to identify the biased data. So, effectively we spent much more CPU time.

In Fig. 1 we show our data multiplied by  $n^{0.98}$  as a function of  $n$  in a log-log plot. Due to universality, the curves should become parallel, straight lines of slope  $0.98 - \tilde{t}$  for large  $n$ . Because of the corrections to scaling, they are curved, fortunately in the opposite way. So, the bond percolation data imply  $\tilde{t} \leq 0.98$ , while the site percolation data imply  $0.96 \leq \tilde{t}$ , excluding therefore already both conjectures  $91/96$ <sup>(2)</sup> and  $1$ <sup>(3)</sup>.

We tried several fits based on the minimalization of the square distance from the individual data points. A logarithmic prefactor fits very badly and can be excluded. Fitting with one power-law correction gives leading exponents for site and bond percolation that differ by at least 1%.

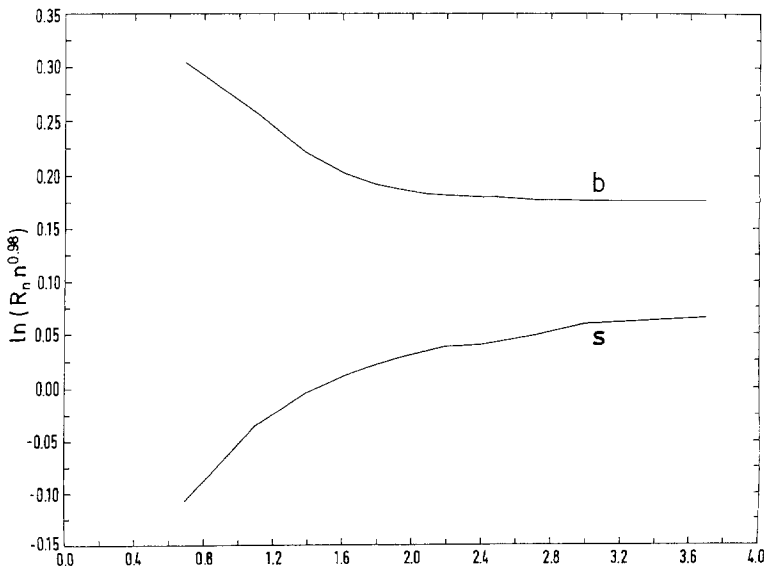


Fig. 1. Log-log plot of  $R_n n^{0.98}$  against  $n$  for (b) bond and (s) site percolations. The bond percolation data are multiplied by a factor 2.5 to make it possible to show all data in one figure.

Fitting a logarithmic correction gives about the same result. The precision of our data should, however, allow for a determination of  $\tilde{\nu}$  better than 1%.

To better understand the above problem, we explicitly imposed that in the relation

$$R_n = n^{-\tilde{\nu}}(c_1 + c_2 n^{-\omega}) \tag{5}$$

not only  $\tilde{\nu}$ , but also the leading correction exponent  $\omega$  should be universal ( $c_1$  and  $c_2$  are nonuniversal constants). So we plotted  $R_n n^{\tilde{\nu}}$  against  $n^{-\omega}$  for different choices of  $\tilde{\nu}$  and  $\omega$  and tried to find  $\tilde{\nu}$  and  $\omega$  such that both site and bond percolation data fall on straight lines. This is not possible, reflecting the problem found before. For site percolation a straight line can be obtained for reasonable values of  $\omega$  ( $\approx 1.2$ ). For bond percolation the data are never aligned and for the parameters for which site percolation data are straight the bond percolation data lie on an S-shaped curve. In Fig. 2 we show such a situation. It becomes clear that higher order corrections play an important role in bond percolation. Only for sizes  $n \geq 7$  does the leading correction seem sufficient, since the data seem to follow a straight asymptotic line (see Fig. 2).

We sought exponents for which the data are reasonably consistent with the scenario of Fig. 2, i.e., straight lines for all site percolation data and for bond percolation with  $n \geq 7$ . We found

$$\tilde{\nu} = 0.9745 \pm 0.0015 \quad \text{and} \quad \omega = 1.2 \pm 0.2$$

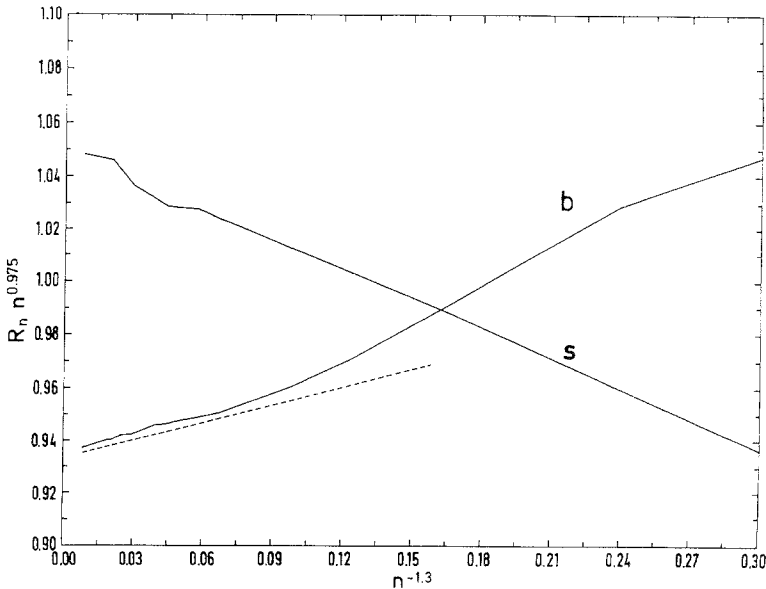


Fig. 2. Plot of  $R_n n^{0.975}$  against  $n^{-1.3}$  for (b) bond and (s) site percolation. The bond percolation data are multiplied by two to make it possible to show all data in the same figure.

The value of  $\tilde{\tau}$  falls within the error bars of all previous estimates<sup>(4,5)</sup> and its error bar is several times smaller than previous error bars.

Besides the very precise calculation of the dynamical exponent of 2D percolation, which excludes the two conjectures 91/96 and 1, we estimated the leading correction to the scaling exponent and found that in the case of bond percolation, sizes  $n < 7$  are dominated by even higher corrections. We discovered that random number generation is the most critical numerical difficulty and we believe that for the future of high-speed computation it is of fundamental importance to find generators that have no pathologies when one has sequences of the order  $10^{11}$ . Our calculation would not have been possible without the special-purpose computer PERCOLA; future work calculating  $\tilde{\tau}$  and  $\tilde{\nu}$  in three and four dimensions on PERCOLA is planned.

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